International Journal of Theoretical Physics, Vol. 28, No. 2, 1989

Asymptotic Procedure for Solving Boundary Value Problems for Singularly Perturbed Linear Impulsive Systems

D. D. Bainov,[1] M. A. Hekimova,[1] and V. M. Veliov[1]

Received August 12, 1988

A justification is given of an asymptotic method for solving a boundary value problem for a linear singularly perturbed impulsive system of differential equations with fast and slow variables.

1. INTRODUCTION

Singularly perturbed systems often arise in mathematical modeling due to the presence of "parasitic" parameters such as small time constants, masses, capacities, etc., multiplying some of the time derivatives [see, e.g., the survey by Kokotović]. The method of the boundary-layer functions (BLFM) (Vasileva and Butuzov, 1973) is a powerful tool for the alleviation of the high dimensionality and the ill-conditioning of such systems. The main aim of this paper is to show that an appropriate modification of BLFM is applicable to singularly perturbed impulsive systems.

Impulsive differential equations represent an effective mathematical apparatus for the investigation of real processes which during their evolution are subject to short-time perturbations. The study of these equations began with the work of Mil'man and Myshkis (1960) and has been extended in various directions related to their applications in physics, biology, radio engineering, automatic control, etc. Periodic singularly perturbed impulsive systems have been investigated by Hekimova and Bainov (1985, 1986).

In this paper we consider a boundary value problem for a linear singularly perturbed system containing stable and unstable "fast" subsystems. The boundary conditions and the impulses (acting at fixed moments

[1]Department of Mathematics, Plovidiv University, Plovidiv, Bulgaria, and Institute of Mathematics, Bulgarian Academy of Sciences, 1090 Sofia, P.O. Box 373, Bulgaria.

0020-7748/89/0200-0209$06.00/0

of time) depend linearly on the state. Problems of this type arise in the treatment of optimal control problems for singularly perturbed systems (without impulses) when the objective function contains extra-integral terms. The mathematical formulation of such a problem is given in Section 2 together with its reduction to the boundary value problem studied in the following two sections. Section 3 presents the formal asymptotic expansions composing the solution and Section 4 is devoted to their convergence.

2. STATEMENT OF THE PROBLEM

Consider the following optimal control problem:

$$\sum_{i=1}^{l} \langle 0.5\, P_i x(t_i) + p_i, x(t_i)\rangle + \varepsilon \sum_{i=1}^{l} \langle 0.5\, S_i \psi(t_i) + s_i, \psi(t_i)\rangle$$
$$+ \int_0^T (\langle 0.5\, M(t)x(t) + m(t), x(t)\rangle + \langle 0.5\, N(t)u(t) + n(t), u(t)\rangle)\, dt \to \min \tag{1}$$

$$\dot{x} = F_{11}(t)x + F_{12}(t)\psi + E_1(t)u + g_1(t), \qquad x(0) = x_0$$
$$\varepsilon\dot{y} = F_{21}(t)x + F_{22}(t)\psi + g_2(t), \qquad \psi(0) = \psi_0$$

where $(x, \psi) \in \mathbb{R}^{m_1 \times m_2}$ is the state vector, $u \in \mathbb{R}^r$ is the control parameter, $F_{ij}, g_i, E_1, M, m, N$, and n are continuous matrix- or vector-valued functions, and P_k, ρ_k, S_k, and s_k are matrices or vectors with appropriate dimensions, $i, j = 1, 2$, $k = \overline{1, l}$. The final time T, the initial conditions (x_0, ψ_0), and the moments $t_1, \ldots, t_l \in (0, T]$, $t_l = T$, of trajectory penalization are fixed. Here ε is a "small" positive parameter which represents the singular perturbation.

If the matrices P_i, S_i, and $M(t)$ are nonnegatively definite and $N(t)$ is positively definite ($i = \overline{1, l}$, $t \in [0, T]$), then there exists a unique solution $(u(\cdot), x(\cdot), \psi(\cdot))$ of the above problem in the control space $L_2^r[0, T]$. Moreover, this solution can be represented as

$$u(t) = N^{-1}(t)(E_1^*(t)y(t) + n(t)) \tag{2}$$

where $y(\cdot)$ together with an appropriate function $\tilde{\eta}(\cdot)$ satisfies the adjoint equation

$$\dot{y} = -F_u^*(t)y - \frac{1}{\varepsilon} F_{21}^*(t)\tilde{\eta} + M(t)x(t) + m(t), \qquad y(T) = 0$$
$$\dot{\tilde{\eta}} = F_{12}^*(t)y - \frac{1}{\varepsilon} F_{22}^*(t)\tilde{\eta} \qquad \tilde{\eta}(T) = 0 \tag{3}$$

and the transversality conditions

$$\begin{aligned}\Delta y(t_i) &= P_i x(t_i) + p_i \\ \Delta \tilde{\eta}(t_i) &= \varepsilon(S_i \psi(t_i) + s_i), \qquad i = \overline{1, l}\end{aligned} \tag{4}$$

This means that $y(\cdot)$ and $\tilde{\eta}(\cdot)$ are differentiable and satisfy (3) on each interval (t_{i-1}, t_i), $i = \overline{1, l}$ $(t_0 = 0)$, and (4) at the points t_i, where $\Delta f(t)$ denotes the difference $f(t+0) - f(t-0)$. Replacing $\tilde{\eta}$ by $\eta = \tilde{\eta}/\varepsilon$ and substituting (2) in the differential equation in (1), we obtain a boundary value problem with impulses for the optimal trajectory $(x(\cdot), \psi(\cdot))$. Using (4) for $i = l$ as a terminal condition, we come to a problem of the following type:

$$\begin{aligned}\dot{x} &= A_1(t)x + B_1(t)y + C_1(t)\psi + D_1(t)\eta + f_1(t), \qquad t \neq t_i \\ \dot{y} &= A_2(t)x + B_2(t)y + C_2(t)\psi + D_2(t)\eta + f_2(t), \qquad t \neq t_i \\ \varepsilon\dot{\psi} &= A_3(t)x + B_3(t)y + C_3(t)\psi + f_3(t), \qquad t \neq t_i \\ \varepsilon\dot{\eta} &= A_4(t)x + B_4(t)y + D_4(t)\eta + f_4(t), \qquad t \neq t_i\end{aligned} \tag{5}$$

$$\begin{aligned}\Delta y(t_i) &= P_i x(t_i) + p_i \\ \Delta \eta(t_i) &= S_i \psi(t_i) + s_i\end{aligned} \tag{6}$$

$$\begin{aligned}&x(0, \varepsilon) = x_0, \qquad \psi(0, \varepsilon) = \psi_0 \\ &y(T, \varepsilon) = P_T x(T) + p_T, \qquad \eta(T, \varepsilon) = S_T \psi(T) + s_T\end{aligned} \tag{7}$$

Henceforth we suppose more generally that $x \in \mathbb{R}^{m_1}$, $y \in \mathbb{R}^{m_2}$, $\psi \in \mathbb{R}^{m_3}$, $\eta \in \mathbb{R}^{m_4}$ and all matrix- and vector-valued functions in (5)-(7) are of appropriate dimensions. As above, $t_1, \ldots, t_p \in (0, T)$, x_0, ψ_0 are fixed initial conditions, and $\varepsilon > 0$ is a "small" parameter.

We shall use the following notations and assumptions.

For $x = (x_1, \ldots, x_q) \in \mathbb{R}^q$ and $A = \{a_{ij}\} \in \mathbb{R}^{q \times r}$ we set

$$|x| = \max_{1 \le i \le q} |x_i|, \qquad |A| = \max_{1 \le i \le q} \sum_{j=1}^{r} |a_{ij}|$$

By $\tilde{C}_q[0, T]$ we denote the space of all q-dimensional functions $w(\cdot)$ such that $w(\cdot)$ is continuous at each $t \in [0, T] \setminus \{t_i, \ldots, t_p\}$ and there exist finite $w(t_i + 0)$ and $w(t_i - 0) = w(t_i)$, $i = 1, \ldots, p$. A norm in $\tilde{C}_q$ is defined by $\sup\{|w(t)|, t \in [0, T]\}$.

Assume that:

A1. For some $n \ge 0$ the matrix-valued functions A_i, B_i, $i = \overline{1, 4}$, C_i, $i = \overline{1, 3}$, and D_i, $i = 1, 2, 4$, are $(n+1)$-times continuously differentiable in $[0, T]$.

A2. The eigenvalues of the matrix $C_3(t)$ have negative real parts, i.e., $\operatorname{Re}\sigma(C_3(t))<0$, and the eigenvalues of the matrix $D_4(t)$ have positive real parts, i.e., $\operatorname{Re}\sigma(D_4(t))>0$.

A3. The functions $f_j(t)\in \tilde{c}_{m_j}[0,T)(j=\overline{1,4})$.

A4. The matrices $E+P_i$, $E+S_i$ $(i=\overline{1,p})$ are nondegenerate.

Observe that if $\operatorname{Re}\sigma(F_{22}(t))<0$ and all the functions in (1) are sufficiently smooth, then our assumptions concerning the optimal control problem (1) imply the properties A1–A4 of the corresponding boundary value problem (5)–(7).

Consider the homogeneous system

$$\begin{aligned}\dot{v}_1&=\bar{A}_1(t)v_1+\bar{B}_1(t)v_2\\ \dot{v}_2&=\bar{A}_2(t)v_1+\bar{B}_2(t)v_2\\ \Delta v_2(t_i)&=P_i v_1(t_i)\end{aligned}\tag{8}$$

where

$$\begin{aligned}\bar{A}_j(t)&=A_j(t)-C_j(t)C_3^{-1}(t)A_3(t)-D_j(t)D_4^{-1}(t)A_4(t)\\ \bar{B}_j(t)&=B_j(t)-C_j(t)C_3^{-1}(t)B_3(t)-D_j(t)D_4^{-1}(t)B_4(t)\qquad (j=1,2)\end{aligned}$$

Denote by $V(t,s)$ $[V(s,s)=E]$ the fundamental matrix of system (8) and write it down in the form

$$V(t,s)=\begin{pmatrix}V_{11}(t,s) & V_{12}(t,s)\\ V_{21}(t,s) & V_{22}(t,s)\end{pmatrix}$$

where $V_{ij}(t,s)$ is an $(m_i\times m_j)$ matrix $(i,j=1,2)$.

We shall use the following assumption as well.

A5. The matrix $P_T V_{12}(T,0)-V_{22}(T,0)$ is nondegenerate.

We shall note that the matrix $V(t,s)$ can be represented in the form

$$V(t,s)=\begin{cases} v(t,s), & t_i<s\le t\le t_{i+1}\quad (i=\overline{0,p})\\[2mm] v(t,t_i)\left[\prod\limits_{j=i}^{k+1}(E+\bar{P}_j)v(t_j,t_{j-1})\right](E+\bar{P}_k)v(t_k,s) & \\ \quad t_{k-1}<s\le t_k<t_i<t\le t_{i+1} & (k=\overline{1,p-1};\quad i=\overline{2,p};\quad k<i)\\[2mm] v(t,t_i)(E+\bar{P}_i)v(t_i,s), & t_{i-1}<s\le t_i<t\le t_{i+1}\quad (i=\overline{1,p})\end{cases}$$

where $t_0=0$, $t_{p+1}=T$, $\bar{P}_i=\left(\begin{smallmatrix}0&0\\P_i&0\end{smallmatrix}\right)$, and by $v(t,s)$ we have denoted the fundamental matrix of the system without impulses corresponding to system (8).

In Section 4 we shall prove that problem (5)–(7) has a solution $r(t,\varepsilon)=(x(t,\varepsilon),y(t,\varepsilon),\psi(t,\varepsilon),\eta(t,\varepsilon))$. Moreover, we shall find an approximation

$R_n(t, \varepsilon)$ of the solution such that

$$\|R_n(\cdot, \varepsilon) - r(\cdot, \varepsilon)\| \leq C\varepsilon^{n+1}$$

The approximation $R_n(t, \varepsilon)$ will be constructed in the form

$$R_n(t, \varepsilon) = \sum_{k=0}^{n} \varepsilon^k [\bar{r}_k(t) + \Pi_k^{(i)} r(\tau_i) + Q_k^{(i)} r(\sigma_i)], \qquad t \in (t_i, t_{i+1}] \qquad (9)$$

where $\bar{r}_k(\cdot)$ is the solution of an appropriate lower order boundary value problem and $\Pi_k^{(i)} r(\cdot)$ and $Q_k^{(i)} r(\cdot)$ are the solutions of appropriate "boundary layer equations" which are also lower dimensional and represented in the "stretched" time scales

$$\tau_i = \frac{t - t_i}{\varepsilon}, \qquad \sigma_i = \frac{t - t_{i+1}}{\varepsilon}, \qquad t \in (t_i, t_{i+1}), \quad i = \overline{0, p} \qquad (10)$$

3. ASYMPTOTIC PRESENTATION

We shall search for a formal asymptotic representation of the solution $r(t, \varepsilon)$ of problem (5)-(7) in the form

$$r(t, \varepsilon) = \bar{r}(t, \varepsilon) + \Pi^{(i)} r(\tau_i, \varepsilon) + Q^{(i)} r(\sigma_i, \varepsilon), \qquad t_i < t < t_{i+1} \qquad (11)$$

where

$$\bar{r}(t, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \bar{r}_k(t), \qquad t \in [0, T] \qquad (12)$$

$$\Pi^{(i)} r(\tau_i, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \Pi_k^{(i)} r(\tau_i), \qquad (i = \overline{0, p})$$

$$Q^{(i)} r(\sigma_i, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k Q_k^{(i)} r(\sigma_i), \qquad (13)$$

where τ_i and σ_i are given by (10).

The coefficients in the expansions (13) are called boundary functions. On them we impose the additional condition

$$\Pi_k^{(i)} r(+\infty) = 0, \qquad Q_k^{(i)} r(-\infty) = 0 \qquad (i = \overline{0, p}) \qquad (14)$$

Setting $\varepsilon = 0$ in (5), we obtain the so-called reduced system

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}_1(t)\bar{x} + \bar{B}_1(t)\bar{y} + \bar{f}_1(t), \qquad t \neq t_i \\ \dot{\bar{y}} &= \bar{A}_2(t)\bar{x} + \bar{B}_2(t)\bar{y} + \bar{f}_2(t), \qquad t \neq t_i \\ \bar{\psi} &= -C_3^{-1}(t)[A_3(t)\bar{x} + B_3(t)\bar{y} + f_3(t)], \qquad t \neq t_i \\ \bar{\eta} &= -D_4^{-1}(t)[A_4(t)\bar{x} + B_4(t)\bar{y} + f_4(t)], \qquad t \neq t_i \end{aligned} \qquad (15)$$

$$\Delta\bar{y}(t_i) = P_i\bar{x}(t_i) + p_i \qquad (16)$$

where we have put

$$\bar{f}_j(t) = f_j(t) - C_j(t)C_3^{-1}(t)f_3(t) - D_j(t)D_4^{-1}(t)f_4(t) \qquad (j = 1, 2)$$

The assumption A5 implies the existence and uniqueness of the solution $(\tilde{x}(t), \tilde{y}(t), \tilde{\psi}(t), \tilde{\eta}(t))$ or (15), (16) satisfying the boundary conditions

$$\tilde{x}(0)=\tilde{x}_0, \qquad \tilde{y}(T)=P_T\tilde{x}(T)+\tilde{p}_T \tag{17}$$

and the relations

$$\Delta\tilde{x}(t_i)=a_i, \qquad \Delta\tilde{y}(t_i)=P_i\tilde{x}(t_i)+b_i \tag{18}$$

for the impulses, where $a_i, \tilde{x}_0, b_i,$ and $\tilde{p}_T$ are arbitrarily fixed vectors. Moreover, this solution can be represented as

$$\begin{aligned}
\tilde{x}(t)&=V_{11}(t,0)\tilde{x}_0+V_{12}(t,0)\tilde{y}_0+\int_0^t[V_{11}(t,s)\bar{f}_1(s)+V_{12}(t,s)\bar{f}_2(s)]\,ds\\
&\quad+\sum_{0<t_\nu<t}V_{11}(t,t_\nu+0)a_\nu+\sum_{0<t_\nu<t}V_{12}(t,t_\nu+0)b_\nu\\
\tilde{y}(t)&=V_{21}(t,0)\tilde{x}_0+V_{22}(t,0)\tilde{y}_0+\int_0^t[V_{21}(t,s)\bar{f}_1(s)+V_{22}(t,s)\bar{f}_2(s)]\,ds\\
&\quad+\sum_{0<t_\nu<t}V_{21}(t,t_\nu+0)+\sum_{0<t_\nu<0}V_{22}(t,t_\nu+0)b_\nu\\
\tilde{\psi}(t)&=-C_3^{-1}(t)[A_3(t)\tilde{x}(t)+B_3(t)\tilde{y}(t)+f_3(t)]\\
\tilde{\eta}(t)&=-D_4^{-1}(t)[A_4(t)\tilde{x}(t)+B_4(t)\tilde{y}(t)+f_4(t)]
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
\tilde{y}_0=&-[P_TV_{12}(T,0)-V_{22}(T,0)]^{-1}\{[P_TV_{11}(T,0)-V_{21}(T,0)]\tilde{x}_0\\
&+\int_0^T\{[P_TV_{11}(T,s)-V_{11}(T,s)]\bar{f}_1(s)\\
&+[P_TV_{12}(T,s)-V_{22}(T,s)]\bar{f}_2(s)\}\,ds\\
&+\sum_{\nu=1}^{p}\{[P_TV_{11}(T,t_\nu+0)-V_{21}(T,t_\nu+0)]a_\nu+[P_TV_{12}(T,t_\nu+0)\\
&-V_{22}(T,t_\nu+0)]b_\nu\}+p_T\}
\end{aligned}$$

For the sake of convenience, we shall also use the notations

$$z=\begin{pmatrix}x\\y\end{pmatrix},\qquad \zeta=\begin{pmatrix}\psi\\\eta\end{pmatrix},\qquad A(t)=\begin{pmatrix}A_1(t)&B_1(t)\\A_2(t)&B_2(t)\end{pmatrix}$$

$$C(t)=\begin{pmatrix}C_1(t)&D_1(t)\\C_2(t)&D_2(t)\end{pmatrix},\qquad B(t)=\begin{pmatrix}A_3(t)&B_3(t)\\A_4(t)&B_4(t)\end{pmatrix}$$

$$D(t)=\begin{pmatrix}C_3(t)&0\\0&D_4(t)\end{pmatrix},\qquad f(t)=\begin{pmatrix}f_1(t)\\f_2(t)\end{pmatrix},\qquad F(t)=\begin{pmatrix}f_3(t)\\f_4(t)\end{pmatrix}$$

$$\bar{A}(t)=\begin{pmatrix}\bar{A}_1(t)&\bar{B}_1(t)\\\bar{A}_2(t)&\bar{B}_2(t)\end{pmatrix},\qquad \bar{f}(t)=\begin{pmatrix}\bar{f}_1(t)\\\bar{f}_2(t)\end{pmatrix},\qquad \bar{F}(t)=-D^{-1}(t)F(t)$$

Substitute (11) into (5)-(7), representing t in the coefficients of system (1) in front of the functions $\Pi_k^{(i)}r(\tau_i)$ in the form $t=t_i+\varepsilon\tau_i$ and in front of the functions $Q_k^{(i)}r(\sigma_i)$ in the form $t=t_{i+1}+\varepsilon\sigma_i$. Afterward we expand the functions $A_j(t_i+\varepsilon\tau_i)$, $A_j(t_{i+1}+\varepsilon\sigma_i)$, $B_j(t_i+\varepsilon\tau_i)$, $B_j(t_{i+1}+\varepsilon\sigma_i)$ $(j=\overline{1,4})$, $C_j(t_i+\varepsilon\tau_i)$, $C_j(t_{i+1}+\varepsilon\sigma_i)(j=\overline{1,3})$, and $D_j(t_i+\varepsilon\tau_i)$, $D_j(t_{i+1}+\varepsilon\tau_i)$ $(j=1,2,4)$ in series by powers of ε and equate the coefficients at the equal powers of ε (separately those depending on t, τ_i, τ_i). Thus we obtain the systems from which we can determine the coefficients of the expansions (12), (13).

For the boundary functions for $k=0$ we obtain the systems

$$\frac{d\Pi_0^{(i)}z}{d\tau_i}=0, \qquad \frac{dQ_0^{(i)}z}{d\sigma_i}=0 \tag{20}$$

$$\frac{d\Pi_0^{(i)}\psi}{d\tau_i}=C_3(t_i)\Pi_0^{(i)}\psi \tag{21}$$

$$\frac{d\Pi_0^{(i)}\eta}{d\tau_i}=D_4(t_i)\Pi_0^{(i)}\eta \tag{22}$$

$$\frac{dQ_0^{(i)}\psi}{d\sigma_i}=C_3(t_{i+1})Q_0^{(i)}\psi \tag{23}$$

$$\frac{dQ_0^{(i)}\eta}{d\sigma_i}=D_4(t_{i+1})Q_0^{(i)}\eta \tag{24}$$

Since $\Pi_0^{(i)}z(0)=\Pi_0^{(i)}\eta(0)=Q_0^{(i)}z(0)=Q_0^{(i)}\psi(0)=0$, from (20), (22), (23), in view of (14), we obtain

$$\Pi_0^{(i)}z(\tau_i)=\Pi_0^{(i)}\eta(\tau_i)=Q_0^{(i)}z(\sigma_i)=Q_0^{(i)}\psi(\sigma_i)=0 \qquad (i=\overline{0,p}) \tag{25}$$

For the function $\bar{r}_0(t)=(\bar{z}_0(t),\bar{\zeta}_0(t))$, applying (14) and (25), we obtain the system

$$\begin{aligned} &\dot{\bar{z}}_0=\bar{A}(t)\bar{z}_0+\bar{f}(t), && t\neq t_i \\ &\bar{\zeta}_0=\bar{B}(t)\bar{z}_0+\bar{F}(t), && t\neq t_i \\ &\Delta\bar{x}_0(t_i)=0, && \Delta\bar{y}_0(t_i)=P_i\bar{x}_0(t_i)+p_i \end{aligned} \tag{36}$$

with boundary conditions

$$\bar{x}_0(0)=x_0, \qquad \bar{y}_0(T)=P_T\bar{x}_0(T)+p_T \tag{27}$$

Problem (26), (27) has a unique solution $(\bar{z}_0(t),\bar{\zeta}_0(t))$, since it coincides with problem (15), (17), (18) for $a_i=0$, $b_i=p_i$, $\tilde{x}_0=x_0$, $\tilde{p}_T=p_T$.

For the initial conditions of the functions $\Pi_0^{(i)}\psi(\tau_i)$ and $Q_0^{(i)}\eta(\sigma_i)$ we obtain

$$\Pi_0^{(0)}\psi(0)=\psi_i-\bar{\psi}_i(0), \qquad \Pi_0^{(i)}\psi(0)=-\Delta\bar{\psi}_0(t_i) \qquad (i=\overline{1,p}) \tag{28}$$

$$\begin{aligned} &Q_0^{(i)}\eta(0)=\Delta\bar{\eta}_0(t_{i+1})-S_{i+1}\bar{\psi}_0(t_{i+1})-s_i \qquad (i=\overline{0,p-1}) \\ &Q_0^{(p)}\eta(0)=S_T\bar{\psi}_0(T)-\bar{\eta}_0(T)+s_T \end{aligned} \tag{29}$$

Once we have solved problem (26), (27), the initial conditions (28), (29) are completely determined.

We solve systems (21), (24) with initial conditions respectively (28), (29) and obtain ($i=\overline{0,p}$)

$$\begin{aligned} \Pi_0^{(i)}\psi(\tau_i) &= \exp[C_3(t_i)\tau_i]\,\Pi_0^{(i)}\psi(0), \qquad \tau_i \geq 0 \\ Q_0^{(i)}\eta(\sigma_i) &= \exp[D_4(t_{i+1})\sigma_i]\,Q_0^{(i)}\eta(0), \qquad \sigma_i \leq 0 \end{aligned} \tag{30}$$

Thus, all functions $\bar{r}_0(t)$, $\Pi_0^{(i)}(\tau_i)$, and $Q_0^{(i)}r(\sigma_i)$ ($i=\overline{0,p}$) are completely determined.

In order to find $\bar{r}_k(t)$, $\Pi_k^{(i)}r(\tau_i)$, and $Q_k^{(i)}r(\sigma_i)$ ($i=\overline{0,p}$) for $1\leq k\leq n$ we proceed analogously.

For the function $\bar{r}_k(t)=(\bar{z}_k(t),\bar{\zeta}_k(t))$ we obtain the system

$$\begin{aligned} \dot{\bar{z}}_k &= \bar{A}(t)\bar{z}_k + C(t)D^{-1}(t)\dot{\bar{\zeta}}_{k-1}, \qquad t\neq t_i \\ \bar{\zeta}_k &= -D^{-1}(t)B(t)\bar{z}_k + D^{-1}(t)\dot{\bar{\zeta}}_{k-1}, \qquad t\neq t_i \end{aligned} \tag{31}$$

$$\Delta\bar{x}_k(t_i) = a_k^{(i)}, \qquad \Delta\bar{y}_k(t_i) = P_i\bar{x}_k(t_i) + b_k^{(i)} \tag{32}$$

where we have put

$$\begin{aligned} a_k^{(i)} &= -\Pi_k^{(i)}x(0) + Q_k^{(i-t)}x(0) \\ b_k^{(i)} &= P_iQ_k^{(i-1)}x(0) - \Pi_k^{(i)}y(0) + Q_k^{(i-1)}y(0) \end{aligned}$$

with boundary conditions

$$\begin{aligned} \bar{x}_k(0) &= -\Pi_k^{(0)}x(0) \\ \bar{y}_k(T) &= P_T[\bar{x}_k(T) + Q_k^{(p)}x(0)] - Q_k^{(p)}y(0) \end{aligned} \tag{33}$$

For the boundary functions we find the systems

$$\frac{d\Pi_k^{(i)}z}{d\tau_i} = T_k^{(i)}(\tau_i), \qquad \frac{dQ_k^{(i)}z}{d\sigma_i} = G_k^{(i)}(\sigma_i) \qquad (i=\overline{0,p}) \tag{34}$$

$$\frac{d\Pi_k^{(i)}\zeta}{d\tau_i} = D(t_i)\Pi_k^{(i)}\zeta + B(t_i)\Pi_k^{(i)}z + L_k^{(i)}(\tau_i) \qquad (i=\overline{0,p}) \tag{35}$$

$$\frac{dQ_k^{(i)}\zeta}{d\sigma_i} = D(t_{i+1})Q_k^{(i)}\zeta + B(t_{i+1})Q_k^{(i)}z + H_k^{(i)}(\sigma_i) \qquad (i=\overline{0,p}) \tag{36}$$

where ($i=\overline{0,p}$)

$$T_k^{(i)}(\tau_i) = \sum_{s=0}^{k-1}\frac{\tau_i^s}{s!}\left[\frac{d^sA(t_i)}{dt^s}\Pi_{k-s-1}^{(i)}z(\tau_i) + \frac{d^sC(t_i)}{dt^s}\Pi_{k-s-1}^{(i)}\zeta(\tau_i)\right]$$

$$G_k^{(i)}(\sigma_i) = \sum_{s=0}^{k-1}\frac{\sigma_i^s}{s!}\left[\frac{d^sA(t_{i+1})}{dt^s}Q_{k-s-1}^{(i)}z(\sigma_i) + \frac{d^sC(t_{i+1})}{dt^s}Q_{k-s-1}^{(i)}\zeta(\sigma_i)\right]$$

$$L_k^{(i)}(\tau_i) = \sum_{s=1}^{k}\frac{\tau_i^s}{s!}\left[\frac{d^sB(t_i)}{dt^s}\Pi_{k-s}^{(i)}z(\tau_i) + \frac{d^sD(t_i)}{dt^s}\Pi_{k-s}^{(i)}\zeta(\tau_i)\right]$$

$$H_k^{(i)}(\sigma_i)=\sum_{s=1}^{k}\frac{\sigma_i^s}{s!}\left[\frac{d^sB(t_{i+1})}{dt^s}Q_{k-s}^{(i)}z(\sigma_i)+\frac{d^sD(t_{i+1})}{dt^s}Q_{k-s}^{(i)}\zeta(\sigma_i)\right]$$

with initial conditions

$$\begin{aligned}
&\Pi_k^{(0)}\psi(0)=-\bar{\psi}_k(0)\\
&\Pi_k^{(i)}\psi(0)=-\Delta\bar{\psi}_k(t_i)+Q_k^{(i-1)}\psi(0)\qquad (i=\overline{1,p})\\
&Q_k^{(i)}\eta(0)=\Delta\bar{\eta}_k(t_{i+1})+\Pi_k^{(i+1)}\eta(0)\\
&\qquad\qquad -S_{i+1}[\bar{\psi}_k(t_{i+1})+Q_k^{(i)}\psi(0)]\qquad (i=\overline{0,p-1})\\
&Q_k^{(p)}\eta(0)=S_T(\bar{\psi}_k(T)+Q_k^{(p)}\psi(0))-\bar{\eta}_k(T)
\end{aligned}\tag{37}$$

and the initial conditions for the functions $\Pi_k^{(i)}z(\tau_i)$, $\Pi_k^{(i)}\eta(\tau_i)$, $Q_k^{(i)}\psi(\sigma_i)$, and $Q_k^{(i)}z(\sigma_i)$ are appropriately chosen according to condition (14).

First we find successively the functions $\Pi_k^{(i)}z(\tau_i)$, $Q_k^{(i)}z(\sigma_i)$, $\Pi_k^{(i)}\eta(\tau_i)$, and $Q_k^{(i)}\psi(\sigma_i)$ $(i=\overline{0,p})$,

$$\Pi_k^{(i)}z(\tau_i)=-\int_{\tau_i}^{+\infty}T_k^{(i)}(s)\,ds,\qquad Q_k^{(i)}z(\sigma_i)=\int_{-\infty}^{\sigma_i}G_k^{(i)}(s)\,ds\tag{38}$$

$$\begin{aligned}
\Pi_k^{(i)}\eta(\tau_i)&=-\int_{\tau_i}^{+\infty}\exp[D_4(t_i)(\tau_i-s)]\\
&\qquad\times[A_4(t_i)\Pi_k^{(i)}x(s)+B_4(t_i)\Pi_k^{(i)}y(s)+L_{k,2}^{(i)}(s)]\,ds\\
Q_k^{(i)}\psi(\sigma_i)&=\int_{-\infty}^{\sigma_i}\exp[C_3(t_{i+1})(\sigma_i-s)]\\
&\qquad\times[A_3(t_{i+1})Q_k^{(i)}x(s)+B_3(t_{i+1})Q_k^{(i)}y(s)+H_{k,1}^{(i)}(s)]\,ds
\end{aligned}\tag{39}$$

where by $L_{k,l}^{(i)}(s)$ we denoted the vector formed by the last m_4 components of the vector $L_k^{(i)}(s)$ and by $H_{k,1}^{(i)}(s)$ that formed by the first m_3 components of the vector $H_k^{(i)}(s)$.

Making use of (38), we solve problem (31)–(33) analogously to problem (15), (17), (18). Thus, initial conditions (37) are completely determined and we find the functions

$$\begin{aligned}
\Pi_k^{(i)}\psi(\tau_i)&=\exp[C_3(t_i)\tau_i]\,\Pi_k^{(i)}\psi(0)+\int_0^{\tau_i}\exp[C_3(t_i)(\tau_i-s)]\\
&\qquad\times[A_3(t_i)\Pi_k^{(i)}x(s)+B_3(t_i)\Pi_k^{(i)}y(s)+L_{k,1}^{(i)}(s)]\,ds,\qquad \tau_i\ge 0\\
Q_k^{(i)}\eta(\sigma_i)&=\exp[D_4(t_{i+1})\sigma_i]Q_k^{(i)}\eta(0)+\int_0^{\sigma_i}\exp[D_4(t_{i+1})(\sigma_i-s)]\\
&\qquad\times[A_4(t_{i+1})Q_k^{(i)}x(s)+B_4(t_{i+1})Q_k^{(i)}y(s)+H_{k,2}^{(i)}(s)]\,ds,\qquad \sigma_i\le 0
\end{aligned}\tag{40}$$

Thus, the coefficients of the expansions (12) and (13), $\bar{r}_k(t)$, $\Pi_k^{(i)}r(\tau_i)$, and $Q_k^{(i)}r(\sigma_i)$ $(i=\overline{0,p};\ k=\overline{0,n})$, are completely determined.

From condition A2, formulas (30) and (38)-(40), by induction it follows that there exist constants $K_0 > 0$ and $\kappa > 0$ such that the inequalities

$$\begin{aligned} |\Pi_k^{(i)} r(\tau_i)| &\leq K_0 \exp(-\kappa\tau_i), \qquad \tau_i \geq 0 \\ |Q_k^{(i)} r(\tau_i)| &\leq K_0 \exp(\kappa\sigma_i), \qquad \sigma_i \leq 0 \end{aligned} \tag{41}$$

hold for $i = \overline{0, p}$, $\kappa = \overline{0, n}$.

Inequalities (41) imply the convergence of all improper integrals which enter the formulas defining the boundary functions.

Thus, all functions which enter the formal expansion (12) are completely determined. In the next section we shall prove that under the assumptions A1-A5 the partial sums of the series (12) form uniform approximations of the solution of problem (5)-(7).

4. EXISTENCE AND APPROXIMATION OF THE SOLUTION OF (5)-(7)

In this section we shall prove the following theorem.

Theorem 1. Let conditions A1-A5 be satisfied. Then there exist constants $\varepsilon_0 > 0$ and M such that for any $\varepsilon \in (0, \varepsilon_0)$ problem (5)-(7) has a unique solution $r(t, \varepsilon)$ and

$$\|r(t, \varepsilon) - R_n(t, \varepsilon)\| \leq M\varepsilon^{n+1}, \qquad t \in [0, T] \tag{42}$$

The function $R_n(t, \varepsilon)$ in the above estimate is given by (9), and $\bar{r}_k(t)$, $\Pi_k^{(i)} r(\tau_i)$, and $Q_k^{(i)} r(\sigma_i)$ were described in Section 2.

First consider the singularly perturbed system

$$\begin{aligned} \varepsilon\dot{\rho}_1 &= C_3(t)\rho_1 + g_1(t), \qquad t \neq t_i \\ \Delta\rho_1(t_i) &= c_1^{(i)} \\ \varepsilon\dot{\rho}_2 &= D_4(t)\rho_2 + g_2(t), \qquad t \neq t_i \\ \Delta\rho_2(t_i) &= S_i\rho_1(t_i) + c_2^{(i)} \end{aligned} \tag{43}$$

with initial condition

$$\rho_1(0, \varepsilon) = \rho_1^0, \qquad \rho_2(T, \varepsilon) = S_T\rho_1(T) + \rho_2^0 \tag{44}$$

where $\rho_1 \in \mathbb{R}^{m_3}$, $\rho_2 \in \mathbb{R}^{m_4}$.

We shall investigate the problem of existence and estimation of the solution $\rho(t, \varepsilon)$ of the boundary value problem (43), (44) under the following conditions (B):

B1. The matrices $C_3(t)$ and $D_4(t)$ are continuous and satisfy condition A2 for $t \in [0, T]$.

B2. The matrices $E + S_i$ $(i = \overline{1, p})$ are nonsingular.

B3. The functions $g_1(t) \in \tilde{C}_{m_3}[0, T]$, $g_2(t) \in \tilde{C}_{m_4}[0, T]$, and $c_1^{(i)}$, $c_2^{(i)}$ $(i = \overline{1, p})$ are arbitrary m_3, m_4-dimensional vectors.

Denote by $Y_1(t, s, \varepsilon)$ $[Y_1(s, s, \varepsilon) = E]$ the fundamental matrix of the homogeneous system $\varepsilon \dot{\rho}_1 = C_3(t)\rho_1$, $t \in [0, T]$, and by $Y_2(t, s, \varepsilon)$ $[Y_2(s, s, \varepsilon) = E]$ the fundamental matrix of the system $\varepsilon \dot{\rho}_2 = D_4(t)\rho_2$, $t \in [0, T]$.

Flato and Levinson (1963) proved that if condition B1 holds, then the matrices $Y_j(t, s, \varepsilon)$ $(j = 1, 2)$ satisfy for sufficiently small values of the parameter ε the inequalities

$$\begin{aligned} |Y_1(t, s, \varepsilon)| &\le K_1 \exp\left[-\kappa \frac{t-s}{\varepsilon}\right], \qquad 0 \le s \le t \le T \\ |Y_2(t, s, \varepsilon)| &\le K_1 \exp\left[\kappa \frac{t-s}{\varepsilon}\right], \qquad 0 \le t \le s \le T \end{aligned} \tag{45}$$

where $\kappa > 0$ and $K_1 > 0$ are constants.

Lemma 1. Let conditions (B) hold.

Then there exist constants $\varepsilon_0 > 0$ and $K > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, the system (43) with boundary condition (44) has a unique solution $\rho(t, \varepsilon) \in \tilde{C}_m[0, T]$ $[\rho(t, \varepsilon) = \binom{\rho_1(t, \varepsilon)}{\rho_2(t, \varepsilon)}]$. This solution satisfies the inequality

$$\|\rho(t, \varepsilon)\| \le K \max\{\|g\|, |\rho^0|, \max_{1 \le i \le p} |c^{(i)}|\} \tag{46}$$

where

$$m = m_3 + m_4, \qquad g(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}, \qquad \rho^0 = \begin{pmatrix} \rho_1^0 \\ \rho_2^0 \end{pmatrix}, \qquad c^{(i)} = \begin{pmatrix} c_1^{(i)} \\ c_2^{(i)} \end{pmatrix}$$

Proof. From the first equation of (43) we find $\rho_1(t, \varepsilon)$,

$$\rho_1(t, \varepsilon) = Y_1(t, 0, \varepsilon)\rho_1^0 + \frac{1}{\varepsilon} \int_0^t Y_1(t, s, \varepsilon) g_1(s)\, ds + \sum_{0 < t_\nu \le t} Y_1(t, t_\nu, \varepsilon) c_1^{(\nu)} \tag{47}$$

From (45) and (47) for $\|\rho_1(t, \varepsilon)\|$ we obtain the estimate

$$\|\rho_1(t, \varepsilon)\| \le K_3 \max\{\|g_1(t)\|, |\rho_1^0|, \max_{1 \le i \le p} |c_1^{(i)}|\} \tag{48}$$

where $K_3 > 0$ is a constant.

Consider the function

$$\begin{aligned} \rho_2(t, \varepsilon) = Y_2(t, T, \varepsilon)\tilde{\rho}_2^0 &+ \frac{1}{\varepsilon} \int_T^t Y_2(t, s, \varepsilon) g_2(s)\, ds \\ &- \sum_{t < t_\nu < T} Y_2(t, t_\nu, \varepsilon)[S_\nu \rho_1(t_\nu) + c_2^{(\nu)}] \end{aligned} \tag{49}$$

where $\rho_1(t_i)$ $(i = \overline{0, p+1})$ are defined by (47) and $\tilde{\rho}_2^0 = S_T \rho_1(T) + \rho_2^0$.

A straightforward verification shows that the function $\rho(t, \varepsilon) = \binom{\rho_1(t,\varepsilon)}{\rho_2(t,\varepsilon)}$ obtained is a solution of problem (43), (44). Estimate (46) follows from (45), (48), and (49). Lemma 1 is proved. ■

Proof of Theorem 1. In system (5), (6) we carry out a change of the variables by the formulas

$$z = u + Z_n(t, \varepsilon), \qquad \zeta = v + H_n(t, \varepsilon) \tag{50}$$

We obtain the system

$$\begin{aligned} \dot{u} &= \bar{A}(t)u + P(t)[B(t)u + D(t)v] + f_1(t, \varepsilon), \qquad t \neq t_i \\ \varepsilon \dot{v} &= B(t)u + D(t)v + f_2(t, \varepsilon), \qquad t \neq t_i \end{aligned} \tag{51}$$

$$\begin{aligned} \Delta u_1(t_i) &= \alpha_1^{(i)}(\varepsilon), \qquad \Delta u_2(t_i) = P_i u_1(t_i) + \alpha_2^{(i)}(\varepsilon) \\ \Delta v_1(t_i) &= \beta_1^{(i)}(\varepsilon), \qquad \Delta v_2(t_i) = S_i v_1(t_i) + \beta_2^{(i)}(\varepsilon) \end{aligned} \tag{52}$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad (u_j \in \mathbb{R}^{m_j}, j = 1, 2, \quad v_1 \in \mathbb{R}^{m_3}, \quad v_2 \in \mathbb{R}^{m_4})$$

$$P(t) = C(t)D^{-1}(t)$$

$$f_1(t, \varepsilon) = A(t)Z_n(t, \varepsilon) + C(t)H_n(t, \varepsilon) - \frac{dZ_n(t, \varepsilon)}{dt}$$

$$f_2(t, \varepsilon) = B(t)Z_n(t, \varepsilon) + D(t)H_n(t, \varepsilon) - \varepsilon \frac{dH_n(t, \varepsilon)}{dt}$$

$$\begin{aligned} \alpha_1^{(i)}(\varepsilon) &= \sum_{k=0}^{n} \varepsilon^k \Pi_k^{(i-1)} x \left(\frac{t_i - t_{i-1}}{\varepsilon} \right) - \sum_{k=0}^{n} \varepsilon^k Q_k^{(i)} x \left(\frac{t_i - t_{i+1}}{\varepsilon} \right) \\ \alpha_2^{(i)}(\varepsilon) &= P_i \sum_{k=0}^{n} \varepsilon^k \Pi_k^{(i-1)} x \left(\frac{t_i - t_{i-1}}{\varepsilon} \right) + \sum_{k=0}^{n} \varepsilon^k \Pi_k^{(i-1)} y \left(\frac{t_i - t_{i-1}}{\varepsilon} \right) \\ &\quad - \sum_{k=0}^{n} \varepsilon^k Q_k^{(i)} y \left(\frac{t_i - t_{i+1}}{\varepsilon} \right) \\ \beta_1^{(i)}(\varepsilon) &= \sum_{k=0}^{n} \varepsilon^k \Pi_k^{(i-1)} \psi \left(\frac{t_i - t_{i-1}}{\varepsilon} \right) - \sum_{k=0}^{n} \varepsilon^k Q_k^{(i)} \psi \left(\frac{t_i - t_{i+1}}{\varepsilon} \right) \\ \beta_2^{(i)}(\varepsilon) &= S_i \sum_{k=0}^{n} \varepsilon^k \Pi_k^{(i-1)} \psi \left(\frac{t_i - t_{i-1}}{\varepsilon} \right) + \sum_{k=0}^{n} \varepsilon^k \Pi_k^{(i-1)} \eta \left(\frac{t_i - t_{i-1}}{\varepsilon} \right) \\ &\quad - \sum_{k=0}^{n} \varepsilon^k Q_k^{(i)} \eta \left(\frac{t_i - t_{i+1}}{\varepsilon} \right) \end{aligned} \tag{53}$$

For system (51), (52) from (7) and (50) we obtain the boundary conditions

$$u_1(0,\varepsilon)=p_1(\varepsilon),\qquad u_2(T,\varepsilon)=P_T u_1(T)+p_2(\varepsilon)$$
$$v_1(0,\varepsilon)=q_1(\varepsilon),\qquad v_2(T,\varepsilon)=S_T v_1(T)+q_2(\varepsilon) \tag{54}$$

where

$$p_1(\varepsilon)=-\sum_{k=0}^{n}\varepsilon^k Q_k^{(0)}x\left(-\frac{t_1}{\varepsilon}\right),\qquad q_1(\varepsilon)=-\sum_{k=0}^{n}\varepsilon^k Q_k^{(0)}\psi\left(-\frac{t_1}{\varepsilon}\right)$$
$$p_2(\varepsilon)=P_T\sum_{k=0}^{n}\varepsilon^k\Pi_k^{(p)}x\left(\frac{T-t_p}{\varepsilon}\right)-\sum_{k=0}^{n}\varepsilon^k\Pi_k^{(p)}y\left(\frac{T-t_p}{\varepsilon}\right) \tag{55}$$
$$q_2(t)=S_T\sum_{k=0}^{n}\varepsilon^k\Pi_k^{(p)}\psi\left(\frac{T-t_p}{\varepsilon}\right)-\sum_{k=0}^{n}\varepsilon^k\Pi_k^{(p)}\eta\left(\frac{T-t_p}{\varepsilon}\right)$$

From relations (20)-(29), (31)-(37), (41), (53), and (55) it follows that for sufficiently small values of the parameter the following inequalities hold:

$$|P(t)|\le N_0,\qquad |f_2(t,\varepsilon)|\le N_1\varepsilon^{n+1},\qquad t\in[0,T]$$
$$|f_1(t,\varepsilon)|\le N_2\varepsilon^n\left[\exp\left(-\kappa\frac{t-t_i}{\varepsilon}\right)\right.$$
$$\left.+\exp\left(\kappa\frac{t-t_{i+1}}{\varepsilon}\right)\right],\qquad t_i<t\le t_{i+1},\quad i=\overline{0,p} \tag{56}$$
$$|\alpha_k^{(i)}(\varepsilon)|\le N_3\varepsilon^{n+1},\qquad |\beta_k^{(i)}(\varepsilon)|\le N_4\varepsilon^{n+1},\qquad k=1,2,\quad i=\overline{1,p}$$
$$|p_k(\varepsilon)|\le N_5\varepsilon^{n+1},\qquad |q_k(\varepsilon)|\le N_6\varepsilon^{n+1},\quad k=1,2$$

where N_j $(j=\overline{0,6})$ are positive constants.

Consider the set

$$T_\delta=\{w\colon w\in\tilde{C}_m[0,T],\ \|w\|\le\delta\}$$

where $\delta>0$ is a constant.

For $w\in T_\delta$ from Lemma 1 it follows that the system

$$\varepsilon\dot h=D(t)h+B(t)w+f_2(t,\varepsilon),\qquad t\ne t_i$$
$$\Delta h_1(t_i)=\beta_1^{(i)}(\varepsilon),\qquad \Delta h_2(t_i)=S_i h_1(t_i)+\beta_2^{(i)}(\varepsilon) \tag{57}$$

with initial condition

$$h_1(0,\varepsilon)=q_1(\varepsilon),\qquad h_2(T,\varepsilon)=S_T h_1(T)+q_2(\varepsilon) \tag{58}$$

has a unique solution $h(t,w,\varepsilon)\in\tilde{C}_m[0,T]$.

From relations (46) and (56) it follows that there exist constants $\varepsilon_0 > 0$, $L_0 > 0$, and $L_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ the following inequalities hold:

$$\begin{aligned} \|h(t, w, \varepsilon)\| &\le L_0\|w\| + L_1\varepsilon^{n+1}, \qquad w \in T_\delta \\ \|h(t, w', \varepsilon) - h(t, w'', \varepsilon)\| &\le L_0\|w' - w''\|, \qquad w', w'' \in T_\delta \end{aligned} \tag{59}$$

Consider the operator $\mathscr{C}_\varepsilon$ associating with each function $w \in T_\delta$ the unique solution $\mathscr{C}_\varepsilon w$ of the system

$$\begin{aligned} \tilde{u} &= \bar{A}(t)\tilde{u} + P(t)[B(t)w + D(t)h] + f_1(t, \varepsilon), \qquad t \ne t_i \\ \Delta\tilde{u}_1(t_i) &= \alpha_1^{(i)}(\varepsilon), \qquad \Delta\tilde{u}_2(t_i) = P_i\tilde{u}_1(t_i) + \alpha_2^{(i)}(\varepsilon) \end{aligned} \tag{60}$$

satisfying the boundary condition

$$\tilde{u}_1(0, \varepsilon) = p_1(\varepsilon), \qquad \tilde{u}_2(T, \varepsilon) = P_T\tilde{u}_1(T) + p_2(\varepsilon) \tag{61}$$

Making use of the solution of problem (15), (17), (18), we write down the solution of the boundary value problem (60), (61) in the form

$$\mathscr{C}_\varepsilon w = \int_0^t V(t, s)P(s)[B(s)w(s) + D(s)h(s)]\, ds + G(t, \varepsilon) \tag{62}$$

where

$$G(t, \varepsilon) = \begin{pmatrix} G_1(t, \varepsilon) \\ G_2(t, \varepsilon) \end{pmatrix}$$

$$\begin{aligned} G_1(t, \varepsilon) = {} & V_{11}(t, 0)p_1(\varepsilon) + V_{12}(t, 0)\tilde{u}_2^0 + \int_0^t (V(t, s)f_1(s, \varepsilon))_1\, ds \\ & + \sum_{0<t_\nu<t} V_{11}(t, t_\nu + 0)\alpha_1^{(\nu)}(\varepsilon) + \sum_{0<t_\nu<t} V_{12}(t, t_\nu + 0)\alpha_2^{(\nu)}(\varepsilon) \end{aligned} \tag{63}$$

$$\begin{aligned} G_2(t, \varepsilon) = {} & V_{21}(t, 0)p_1(\varepsilon) + V_{22}(t, 0)\tilde{u}_2^0 + \int_0^t (V(t, s)f_1(s, \varepsilon))_2\, ds \\ & + \sum_{0<t_\nu<t} V_{21}(t, t_\nu + 0)\alpha_1^{(\nu)}(\varepsilon) + \sum_{0<t_\nu<t} V_{22}(t, t_\nu + 0)\alpha_2^{(\nu)}(\varepsilon) \end{aligned} \tag{64}$$

$$\begin{aligned} \tilde{u}_2^0 = {} & -[P_TV_{12}(T, 0) - V_{22}(T, 0)]^{-1} \\ & \times \Big\{ P_TV_{11}(T, 0) - V_{21}(T, 0)]p_1(\varepsilon) \\ & + \int_0^T P_T^* V(T, s)P(s)[C(s)w(s) + D(s)h(s)]\, ds + p_2(\varepsilon) \\ & + \int_0^T P_T^* V(T, s)f_1(s, \varepsilon)\, ds + \sum_{\nu=1}^{p} P_T^* V(T, t_\nu + 0)\alpha^{(\nu)} \Big\} \end{aligned}$$

where by P_T^* we have denoted the matrix $(P_T, -E)$ of dimension $m_2 \times (m_1+m_2)$ and by $\alpha^{(\nu)}$ the vector

$$\begin{pmatrix} \alpha_1^{(\nu)} \\ \alpha_2^{(\nu)} \end{pmatrix}$$

From (62), taking into account that $h(t, w, \varepsilon)$ is a solution of system (57), we obtain

$$\mathscr{C}_\varepsilon w = \varepsilon \int_0^t V(t, \varepsilon)P(s)h(s)\, ds - \int_0^t V(t, s)P(s)f_2(s, \varepsilon)\, ds + G(t, \varepsilon) \tag{65}$$

where by $\dot{h}(t)$ for $t = t_i$ $(i = \overline{1, p})$ we have denoted $h(t_i - 0)$.

We represent the first addend in the right-hand side of (65) in the form

$$\begin{aligned} \varepsilon \int_0^t & V(t, s)P(s)\dot{h}(s)\, ds \\ &= \varepsilon \int_0^{t_i} V(t, s)P(s)\dot{h}(s)\, ds \\ &\quad + \sum_{i=1}^{k-1} \varepsilon \int_{t_k}^{t_{i+1}} V(t, s)P(s)\dot{h}(s) + \varepsilon \int_{t_k}^{t} V(t, s)P(s)\dot{h}(s)\, ds \end{aligned} \tag{66}$$

where $0 < t_i < \cdots t_k < t \leq t_{k+1} \leq T$.

After an integration by parts from (66) we obtain

$$\begin{aligned} \varepsilon \int_0^t & V(t, s)P(s)\dot{h}(s)\, ds \\ &= \varepsilon P(t)h(t) - \varepsilon V(t, 0)P(0)h(0) + \sum_{i=1}^{k} [V(t, t_i)P(t_i) \\ &\quad - V(t, t_i + 0)P(t_i)(E + \bar{S}_i)]h(t_i) \\ &\quad - \varepsilon \sum_{i=1}^{k} V(t, t_i + 0)P(t_i)\beta^{(i)}(\varepsilon) - \varepsilon \int_0^t \frac{\partial(V(t, s)P(s))}{\partial s} h(s)\, ds \end{aligned} \tag{67}$$

where

$$\bar{S}_i = \begin{pmatrix} 0 & 0 \\ S_i & 0 \end{pmatrix}, \qquad \beta^{(i)}(\varepsilon) = \begin{pmatrix} \beta_1^{(i)}(\varepsilon) \\ \beta_2^{(i)}(t) \end{pmatrix}, \qquad i = \overline{1, p}$$

From (67), making use of estimates (56) and the fact that the matrices $V(t, s)$ and $\partial(V(t, s)P(s))/\partial s$ are bounded for $0 \leq s \leq t \leq T$, we obtain

$$\left| \varepsilon \int_0^t V(t, s)P(s)\dot{h}(s)\, ds \right| \leq \varepsilon N_7 \|h\| + \varepsilon N_8 \|w\| + N_9 \varepsilon^{n+2} \tag{68}$$

for $0 \leq t \leq T$, $\varepsilon \in (0, \varepsilon_0)$, $N_j = \text{const}$, $N_j > 0$ $(j = \overline{7, 9})$.

For the second addend of (65) from (56) we obtain

$$\left|\int_0^t V(t, s)P(s)f_2(s, \varepsilon)\, ds\right| \leq N_{10}\varepsilon^{n+1} \tag{69}$$

for $0 \leq t \leq T$, $\varepsilon \in (0, \varepsilon_0)$, $N_{10} = \text{const}$, $N_{10} > 0$.

In order to estimate $G(t, \varepsilon)$, we use once more (56). We find

$$|G(t, \varepsilon)| \leq N_{11}\varepsilon^{n+1} + N_{12}|\tilde{u}_2^0| \tag{70}$$

for $0 \leq t \leq T$, $\varepsilon \in (0, \varepsilon_0)$, N_{11}, $N_{12} = \text{const}$, N_{11}, $N_{12} > 0$.

Finally, we estimate $|\tilde{u}_2^0|$, making use of (64), (56), (68), and (69) for $t = T$. We obtain

$$|\tilde{u}_2^0| \leq N_{13}\varepsilon^{n+1} + \varepsilon N_{14}\|h\| + \varepsilon N_{15}\|w\| \tag{71}$$

for $\varepsilon \in (0, \varepsilon_0)$, $N_j = \text{const}$, $N_j > 0$ $(j = \overline{13, 15})$.

From (65) and (68)-(71) we obtain

$$\|\mathscr{C}_\varepsilon w\| \leq \varepsilon N_{16}\|h\| + \varepsilon N_{17}\|w\| + N_{18}\varepsilon^{n+2} \tag{72}$$

for $w \in T_\delta$, $\varepsilon \in (0, \varepsilon_0)$, $N_j = \text{const}$, $N_j > 0$ $(j = \overline{16, 18})$.

Analogously, we prove the inequality

$$\|\mathscr{C}w' - \mathscr{C}w''\| \leq N\varepsilon\|h' - h''\| + \bar{N}\varepsilon\|w' - w''\| \tag{73}$$

for w', $w'' \in T_\delta$, $h' = h(t, w', \varepsilon)$, $h'' = h(t, w'', \varepsilon)$, $\varepsilon \in (0, \varepsilon_0)$, N, $\bar{N} = \text{const}$.

From estimates (59), (72), and (73) it follows that for sufficiently small values of the parameter ε the operator $\mathscr{C}_\varepsilon$ is a contractive operator in T_δ. Denote by $u(t, \varepsilon)$ its unique fixed point. Then the function $(u(t, \varepsilon), v(t, \varepsilon) = h(t, u(t, \varepsilon)\varepsilon))$ is the unique solution of boundary value problem (51), (52), (54) for $\varepsilon \in (0, \varepsilon_0)$.

Put $\delta = C\varepsilon$, where the constant $C > 0$ is sufficiently large but fixed. From the first equality of (59) and from (72) we obtain the estimate

$$\|u(t, \varepsilon)\| \leq M\varepsilon^{n+1}, \qquad \|v(t, \varepsilon)\| \leq M\varepsilon^{n+1} \tag{74}$$

From (74) in view of (50) we obtain the assertion of Theorem 1.

REFERENCES

Flato, L., and Levinson, N. (1963). Periodic solutions of singularly perturbed systems, *Journal of Rational Mechanics and Analysis*, **4**, 943-950.

Hekimova, M. A., and Bainov, D. D. (1985). Periodic solutions of singularly perturbed systems of differential equations with impulse effect, *ZAMP*, **36**, 520-537.

Hekimova, M. A., and Bainov, D. D. (1986). A method for asymptotic integration of a periodic singularly perturbed systems with impulses and its application in the theory of optimal control, *Journal de Mathématiques pures et appliquées*, **65**, 1-15.

Kokotović, P. K. (1984). Applications of singular perturbation techniques to control problems, *SIAM Review*, **26**, (4), 501–549.

Mil'man, V. D., and Myshkis, A. D. (1960). On the stability of motion in the presence of impulses, *Siberian Mathematics Journal*, **1**(2), 233–237 (in Russian).

Vasileva, A., and Butuzov, V. (1973). *Asymptotic Expansions of Solutions of Singularly Perturbed Differential Equations*, Nauka, Moscow (in Russian).